

A solution to two party typicality using representation theory of the symmetric group

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1 Abstract

We show that, given a state on a bipartite system AB, the product of the tensor product of the typical projections for the marginal states on A and B and the typical projection for AB can be used to describe the correct asymptotics of the bipartite state itself, its square, marginals and squares of the marginals. Typicality is defined using the representation theory of the symmetric group.

This result has already been proven, but with a different notion of typicality.

2 Introduction

This work is motivated by a recent conjecture by Nicolas Dutil [3, Conjecture 3.2.7], who also gave a solution for the two-party case in the very same work. The conjecture in its original formulation reads as follows.

Conjecture 1 (Multiparty typicality conjecture - Conjecture 3.2.7 in [3]). *Consider n copies of an arbitrary multiparty state $\psi^{C_1 \dots C_m}$. For any fixed $\varepsilon > 0$, $\delta_{\mathcal{T}} > 0$ and n large enough, there exists a state $\Psi^{C_1 \dots C_m}$ which satisfies*

$$\|\Psi^{C_1 \dots C_m} - \psi^{C_1 \dots C_m}\|_1 \leq \nu(\varepsilon) \quad (1)$$

$$\mathrm{tr}\{(\Psi^{\mathcal{T}})^2\} \leq (1 - \mu(\varepsilon))2^{-n(S(\psi_{\mathcal{T}}) - \delta_{\mathcal{T}})} \quad (2)$$

for all non-empty subsets $\mathcal{T} \subset \{1, \dots, m\}$. Here, $\nu(\varepsilon)$ and $\mu(\varepsilon)$ are functions of ε which vanish by choosing arbitrarily small values for ε .

The main obstacle one is confronted with here is the notorious noncommutativity of different tensor products of marginals of the joint state $\Psi^{C_1 \dots C_m}$.

3 Notation

The symbols $\lambda, \lambda', \nu, \nu', \mu, \mu'$ will be used to denote **Young frames**. The set of Young frames with at most $d \in \mathbb{N}$ rows and $n \in \mathbb{N}$ boxes is denoted $YF_{d,n}$.

For a **Young Tableau** T , we write T_{ij} for the entry of T in the i -th row and j -th column.

In the remainder, $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}$ denote **Hilbert spaces** with dimensions d_A, d_B, d . The numbers d_A, d_B will be arbitrary but constant, while d serves as a “dummy”-dimension for intermediate

statements. Dimensions will also be assumed to be strictly larger than one, since otherwise the statements made in this work become trivial.

The symbol $\mathbb{B}^{A \otimes B}$ denotes the product **representation** of $S_n \times S_n$ on $\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}$ induced by the standard representations $\mathbb{B}^A, \mathbb{B}^B$ of the **symmetric group** S_n on $\mathcal{H}_A^{\otimes n}$ and $\mathcal{H}_B^{\otimes n}$.

A representation \mathbb{B}^{AB} of S_n on $\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}$ is then given by the obvious reordering of the standard representation of S_n on $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$.

It holds

$$\mathbb{B}^{AB}(\sigma) = \mathbb{B}^A(\sigma) \otimes \mathbb{B}^B(\sigma) \text{ for all } \sigma \in S_n. \quad (3)$$

The unique complex vector space carrying the irreducible representation of S_n corresponding to a Young Tableau λ will be written F_λ .

The **multiplicity of an irreducible subspace** of \mathbb{B}^X (where $X \in \{A, B, AB, A \otimes B\}$) corresponding to a Young frame λ is denoted m_λ^X .

Projections onto the irreducible subspaces of \mathbb{B}^{AB} are denoted by $P_{\lambda,k}^{AB}$ ($\lambda \in YF_{d_A d_B, n}$, $k \in [m_\lambda^{AB}]$). Implicit here is the choice of a specific set of these, and this set is chosen such that every two different projections are orthogonal (this may be seen as a specific choice of bases for the invariant subspaces U_λ , $\lambda \in YF_{d_A d_B, n}$, of the reordering of the standard representation $U \mapsto U^{\otimes n}$ of the unitary group on $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$). Another constraint will be given by equation (5). Accordingly, projections onto irreducible subspaces of $\mathbb{B}^{A \otimes B}$ get labelled $P_{\mu,i}^A \otimes P_{\nu,j}^B$ ($\mu, \nu \in YF_{d_A, n}, YF_{d_B, n}$, $i, j \in [m_\mu^A], [m_\nu^B]$).

Whenever it feels right, the superscripts A, B, AB will be omitted. To make up for that, in this case, the symbols λ, λ' will only be used for projections on AB , while μ, μ' indicate that a projection on A is being used and ν, ν' are only subscripts for projections on the B -part.

Define, for arbitrary $\mu \in YF_{d_A, n}$, $\nu \in YF_{d_B, n}$, $\lambda \in YF_{d_A d_B, n}$ the projections

$$P_\mu^A := \sum_{i=1}^{m_\mu^A} P_{\mu,i}^A, \quad P_\nu^B := \sum_{j=1}^{m_\nu^B} P_{\nu,j}^B, \quad P_\lambda^{AB} := \sum_{k=1}^{m_\lambda^{AB}} P_{\lambda,k}^{AB}. \quad (4)$$

The choice we just made for the set $\{P_{\lambda,i}^{AB} : \lambda \in YF_{d,n}, i \in [m_\lambda]\}$ gets a little more specific now: We will choose these projections such that each $P_\mu^A \otimes P_\nu^B$ (note that these projections correspond to subspaces which are only *invariant* under the action of \mathbb{B}^{AB}) can, by choosing an appropriate set \mathcal{M} , be written as

$$P_\mu^A \otimes P_\nu^B = \sum_{(\lambda,i) \in \mathcal{M}} P_{\lambda,i}. \quad (5)$$

This is possible due to equation (3). Conversely, it implies that each $P_{\lambda,i}$ obeys the inequality

$$P_{\lambda,i} \leq P_\mu^A \otimes P_\nu^B \quad (6)$$

for exactly one specific choice of $\mu, \nu \in YF_{d_A, n}, YF_{d_B, n}$.

The **set of states** on a Hilbert space \mathcal{H} is written $\mathcal{S}(\mathcal{H})$. The set of **probability distributions** on a finite set \mathbf{X} is denoted $\mathcal{P}(\mathbf{X})$, the cardinality of \mathbf{X} by $|\mathbf{X}|$.

For $\lambda \in YF_{d,n}$, $\bar{\lambda} \in \mathcal{P}([d])$ is defined by $\bar{\lambda}(i) := \lambda_i/n$. If $\rho \in \mathcal{S}(\mathcal{H})$ with $\dim \mathcal{H} = d$ has spectrum $s \in \mathcal{P}([d])$, then it will always be assumed that $s(1) \geq \dots \geq s(d)$ holds and the distance between a spectrum s and a Young frame $\lambda \in YF_{d,n}$ is measured by $\|\bar{\lambda} - s\| := \sum_{i=1}^d |\bar{\lambda}(i) - s(i)|$.

We now define two important entropic quantities, both of which use the **base two logarithm**. Throughout this work, this function will be written \log . Given a finite set \mathbf{X} and two probability distributions $r, s \in \mathcal{P}(\mathbf{X})$, we define the **relative entropy** $D(r||s)$ by

$$D(r||s) := \begin{cases} \sum_{x \in \mathbf{X}} r(x) \log(r(x)/s(x)), & \text{if } s \gg r \\ \infty, & \text{else} \end{cases} \quad (7)$$

In case that $D(r||s) = \infty$, for a positive number $a > 0$, we use the convention $2^{-aD(r||s)} = 0$. The relative entropy is connected to $\|\cdot\|$ by the Pinsker's inequality $D(r||s) \leq \frac{1}{2\ln(2)}\|r - s\|^2$. The **entropy** of $r \in \mathcal{P}(\mathbf{X})$ is defined by the formula

$$H(r) := - \sum_{x \in \mathbf{X}} r(x) \log(r(x)). \quad (8)$$

4 Result

Our result is a collection of estimates concerning the state

$$\Phi := (P_\varepsilon^A \otimes P_\varepsilon^B) P_\delta^{AB} \rho_{AB}^{\otimes n} (P_\varepsilon^A \otimes P_\varepsilon^B). \quad (9)$$

A proper definition is included in the preliminaries of Theorem 1. It is understood that this is our version of the sought-after state $\Psi^{C_1 C_2}$.

The necessary estimates for Φ to fulfill the requirements of the multipartty typicality conjecture, Conjecture 1, are given in inequalities (11), (12) and the right hand inequality of (13). The proof of these inequalities is almost trivial.

The remaining inequalities are stated only for sake of completeness, although especially the left hand inequality in (13) is comparably hard to prove.

Theorem 1. *Let $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ have spectrum r and marginals ρ_A, ρ_B with corresponding spectra r_A, r_B . For every $\varepsilon, \delta > 0$ and $n \in \mathbb{N}$, define the projections*

$$P_\varepsilon^A := \sum_{\mu: \|\bar{\mu} - r_A\| \leq \varepsilon} P_\mu^A, \quad P_\varepsilon^B := \sum_{\nu: \|\bar{\nu} - r_B\| \leq \varepsilon} P_\nu^B \quad \text{and} \quad P_\delta^{AB} := \sum_{\lambda: \|\bar{\lambda} - r\| \leq \delta} P_\lambda^{AB} \quad (10)$$

The dependence of these projections on the parameter n will not be written out in order to simplify notation.

Set $\Phi := (P_\varepsilon^A \otimes P_\varepsilon^B) P_\delta^{AB} \rho_{AB}^{\otimes n} (P_\varepsilon^A \otimes P_\varepsilon^B)$.

There is a function $\gamma : \mathbb{N} \mapsto \mathbb{R}_+$ satisfying $\lim_{n \rightarrow \infty} \gamma(n) = 0$, a function $\varphi : (0, 1/2) \mapsto \mathbb{R}_+$ satisfying $\lim_{\delta \rightarrow 0} \varphi(\delta) = 0$ and an $N \in \mathbb{N}$ such that for all $n \geq N$

$$\text{tr}\{\text{tr}_B\{\Phi\}^2\} \leq 2^{-n(H(r_A) - \gamma(n))} \quad (11)$$

$$\|\Phi - \rho_{AB}^{\otimes n}\|_1 \leq 3 \cdot 2^{-n(\min\{\varepsilon, \delta\}^2 + \gamma(n)/2)} \quad (12)$$

$$2^{-nH(r)} 2^{-n(\varphi(\delta) + \gamma(n))} (1 - 2^{-n\varepsilon^2}) \leq \text{tr}\{\Phi^2\} \leq 2^{-n(H(r) - \gamma(n))} \quad (13)$$

$$\text{tr}\{\Phi\} \geq 1 - 2^{-n(2\min\{\varepsilon, \delta\}^2 + \gamma(n))} \quad (14)$$

$$\|(P_\varepsilon^A \otimes P_\varepsilon^B) \rho_{AB}^{\otimes n} (P_\varepsilon^A \otimes P_\varepsilon^B) - \rho_{AB}^{\otimes n}\|_1 \leq 3 \cdot 2^{-n(\varepsilon^2 + \gamma(n)/2)} \quad (15)$$

$$\text{tr}\{[(P_\varepsilon^A \otimes P_\varepsilon^B) \rho_{AB}^{\otimes n}]^2\} \geq 2^{-n(H(r) + \gamma(n))} (1 - 2^{-2n\varepsilon^2}) \quad (16)$$

$$\text{tr}\{(P_\varepsilon^A \otimes P_\varepsilon^B) \rho_{AB}^{\otimes n}\} \geq 1 - 2^{-n(2\varepsilon^2 + \gamma(n))} \quad (17)$$

Remark 1. *Note that P_δ^{AB} commutes with $P_\varepsilon^A \otimes P_\varepsilon^B$ as well as with $\rho_{AB}^{\otimes n}$.*

We will need a few preliminary results before proving this theorem. First, a few estimates are needed:

With $h(i, j)$ denoting Hook-lengths, the dimensions of the irreducible subspaces of any representation of S_n on $(\mathbb{C}^d)^{\otimes n}$ ($d > 0$) obey the following estimates.

$$\frac{n!}{\prod_{i=1}^n (\lambda_i + d + 1)!} \leq \frac{n!}{\prod_{(i,j) \in \lambda} h(i, j)} = \dim F_\lambda \leq \frac{n!}{\prod_{i=1}^d \lambda_i!} \quad (\lambda \in YF_{d,n}). \quad (18)$$

Also, we are going to employ the following estimate taken from [1], Lemma 2.3:

$$\frac{1}{(n+1)^d} 2^{nH(\bar{\lambda})} \leq \frac{n!}{\prod_{i=1}^d \lambda_i!} \leq 2^{nH(\bar{\lambda})} \quad (\lambda \in YF_{d,n}) \quad (19)$$

as well as, with $\overline{\lambda+d+1}(i) := \frac{1}{n+d(d+1)}(\lambda_i + d + 1)$,

$$\|\bar{\lambda} - \overline{\lambda+d+1}\| = \sum_{i=1}^d \left| \frac{\lambda_i}{n} - \frac{\lambda_i + d + 1}{n + d(d+1)} \right| \quad (20)$$

$$\leq d \cdot \max_{i=1, \dots, d} \left| \frac{\lambda_i}{n} - \frac{\lambda_i + d + 1}{n + d(d+1)} \right| \quad (21)$$

$$= d \cdot \max_{i=1, \dots, d} \left| \frac{\lambda_i d(d+1) - n(d+1)}{n(n + d(d+1))} \right| \quad (22)$$

$$\leq \frac{d(d+1)}{n^2} \cdot \max_{i=1, \dots, d} |\lambda_i d - n| \quad (23)$$

$$\leq \frac{d(d+1)^2}{n} \quad (24)$$

$$\leq \frac{(d+1)^3}{n} \quad (25)$$

$$\leq \frac{8d^3}{n} \quad (26)$$

$$(\text{if } d \geq 2) \leq \frac{d^6}{n} \quad (27)$$

and, at last, Lemma 2.7 from [1]:

Lemma 1. *If, for \mathbf{A} a finite alphabet and $p, q \in \mathfrak{P}(\mathbf{A})$ we have $|p - q| \leq \Theta \leq 1/2$, then*

$$|H(p) - H(q)| \leq -\Theta \log \frac{\Theta}{|\mathbf{A}|}. \quad (28)$$

Combining equations (18) and (19) leads to the estimate

$$\dim F_\lambda \leq 2^{nH(\bar{\lambda})} \quad (\lambda \in YF_{d,n}). \quad (29)$$

Deriving a lower bound on $\dim F_\lambda$ is slightly more involved: Let $n \geq 2d^2$. Then

$$\dim F_\lambda = \frac{n!}{\prod_{i=1}^d (\lambda_i + d + 1)!} \quad (30)$$

$$= \frac{1}{(n + d(d+1)) \cdot \dots \cdot (n+1)} \frac{(n + d(d+1))!}{\prod_{i=1}^d (\lambda_i + d + 1)!} \quad (31)$$

$$\geq \frac{1}{(2n)^{2d^2}} \frac{(n + d(d+1))!}{\prod_{i=1}^d (\lambda_i + d + 1)!} \quad (32)$$

$$\geq \frac{1}{(2n)^{2d^2}} \frac{1}{(2n)^d} 2^{(n+d(d+1))H(\overline{\lambda+d+1})} \quad (33)$$

$$\geq \frac{1}{(2n)^{3d^2}} 2^{nH(\overline{\lambda+d+1})} \quad (34)$$

$$\geq \frac{1}{(2n)^{5d^2}} 2^{n(H(\bar{\lambda}) + \frac{d^6}{n} \log \frac{d^5}{n})} \quad (35)$$

$$= 2^{n(H(\bar{\lambda}) + \frac{d^6}{n} \log \frac{d^5}{n} - \frac{5d^2}{n} \log(2n))}. \quad (36)$$

Set $\gamma_1(n) := -\frac{d^6}{n} \log \frac{d^5}{n} + \frac{5d^2}{n} \log(2n)$, then there is an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have

$$\dim F_\lambda \geq 2^{n(H(\bar{\lambda}) - \gamma_1(n))}. \quad (37)$$

An important step in the application of the representation theory of the symmetric group to quantum information theory was the following theorem:

Theorem 2 ([5]). *For $\lambda \in YF_{d,n}$ and $\sigma \in \mathcal{S}(\mathcal{H})$ ($\dim \mathcal{H} = d$) with spectrum s it holds*

$$\text{tr}\{P_\lambda \sigma^{\otimes n}\} \leq (n+1)^{d(d-1)/2} 2^{-nD(\bar{\lambda}||s)}. \quad (38)$$

Now let us state the first essential ingredient to our theorem.

Lemma 2. *Let $\mu \in YF_{d_A,n}$, $\nu \in YF_{d_B,n}$, $\lambda \in YF_{d_B d_A,n}$.*

1. *If $\|\bar{\lambda} - r\| < \varepsilon$, $\|\bar{\mu} - r_A\| > \delta$, then for $P_{\lambda,i}, P_{\lambda,j}$ ($i, j \in [m_\lambda]$) with $P_{\lambda,i} \leq P_\mu \otimes P_\nu$ and $P_{\lambda,j} \leq P_{\mu'} \otimes P_{\nu'}$ it holds*

$$\text{tr}\{P_{\lambda,i} \rho_{AB}^{\otimes n} P_{\lambda,j} \rho_{AB}^{\otimes n}\} \leq 2^{-n(2\delta^2 + H(r) + \varepsilon \log(\varepsilon d_A d_B) - \gamma_2(n))}. \quad (39)$$

2. *If $\|\bar{\lambda} - r\| > \varepsilon$, then for $P_{\lambda,i}$ and $P_{\lambda,j}$ with $i, j \in [m_\lambda^{AB}]$ it holds*

$$\text{tr}\{P_{\lambda,i} \rho_{AB}^{\otimes n} P_{\lambda,j} \rho_{AB}^{\otimes n}\} \leq 2^{-n(4\varepsilon^2 + H(\bar{\lambda}) - \gamma_7(n))} \quad (40)$$

The function γ_2 is given by $\gamma_2(n) := \frac{(d_A d_B)^2}{n} \log(n+1) + \gamma_1(n)$ and the function γ_7 through the formula $\gamma_7(n) := \frac{2(d_A d_B)^2}{n} \log(n+1) + \gamma_1(n)$.

Remark 2. *The lemma can w.l.o.g. be read with the roles of A and B interchanged.*

Proof. We consider the first statement first. Let us take a look at $P_\lambda \rho_{AB}^{\otimes n}$ first. Observe that the two operators in this product commute. Since $P_\lambda \rho_{AB}^{\otimes n}$ is invariant under permutations, we can write it as

$$P_\lambda \rho_{AB}^{\otimes n} = \sum_{i,j=1}^{m_\lambda} c_{ij} Y_{ij}, \quad (41)$$

where the operators $Y_{ij} \in \mathcal{B}(\mathcal{H}_{AB}^{\otimes l})$ satisfy

$$P_{\lambda,i} Y_{ij} P_{\lambda,j} = Y_{ij}, \quad Y_{ij} Y_{kl} = \delta(j,k) Y_{il} \quad \text{and} \quad Y_{jj} = P_{\lambda,j}. \quad (42)$$

Since $P_\lambda \rho_{AB}^{\otimes n}$ is self-adjoint, we get

$$\sum_{i,j=1}^{m_\lambda} c_{ij} Y_{ij} = \sum_{i,j=1}^{m_\lambda} \bar{c}_{ij} Y_{ij}^\dagger \quad (43)$$

$$= \sum_{i,j=1}^{m_\lambda} \bar{c}_{ij} Y_{ji}, \quad (44)$$

from which it follows that $\bar{c}_{ij} = c_{ji}$. Also, for every $i, j \in [m_\lambda^{AB}]$ we know that

$$c_{ii} Y_{ii} + c_{ij} Y_{ij} + c_{ji} Y_{ji} + c_{jj} Y_{jj} = (P_{\lambda,i} + P_{\lambda,j}) \rho_{AB}^{\otimes n} (P_{\lambda,i} + P_{\lambda,j}) \geq 0. \quad (45)$$

By choosing appropriate bases for $\text{supp}(P_{\lambda,i})$ and $\text{supp}(P_{\lambda,j})$, this translates to the statement

$$\begin{pmatrix} c_{ii} & c_{ij} \\ c_{ji} & c_{jj} \end{pmatrix} \geq 0. \quad (46)$$

This now shows us that $|c_{ij}|^2 \leq |c_{ii}| \cdot |c_{jj}|$ has to hold and that all the c_{ii} , $i = 1, \dots, [m_\lambda^{AB}]$, are nonnegative real numbers. We now prove the promised inequality:

$$\text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}P_{\lambda,j}\rho_{AB}^{\otimes n}\} = \sum_{k,l=1}^{[m_\lambda^{AB}]} c_{kl} \sum_{x,y=1}^{[m_\lambda^{AB}]} c_{xy} \text{tr}\{P_{\lambda,i}Y_{kl}P_{\lambda,j}Y_{xy}\} \quad (47)$$

$$= \sum_{k,l=1}^{[m_\lambda^{AB}]} c_{kl} \sum_{x,y=1}^{[m_\lambda^{AB}]} c_{xy} \text{tr}\{Y_{ii}Y_{kl}Y_{jj}Y_{xy}\} \quad (48)$$

$$= c_{ij}c_{ji} \text{tr}\{Y_{ii}Y_{ij}Y_{jj}Y_{ji}\} \quad (49)$$

$$= |c_{ij}|^2 \text{tr}\{Y_{ii}\} \quad (50)$$

$$= |c_{ij}|^2 \dim(F_\lambda) \quad (51)$$

$$\leq |c_{ii}| \cdot |c_{jj}| \dim(F_\lambda). \quad (52)$$

Observe that

$$|c_{ii}| = c_{ii} \quad (53)$$

$$= \text{tr}\{Y_{ii} \sum_{k,l} c_{kl} Y_{kl}\} / \dim F_\lambda \quad (54)$$

$$= \text{tr}\{P_{\lambda,i}P_\lambda(\rho_{AB}^{\otimes n})^{\otimes n}\} / \dim F_\lambda \quad (55)$$

$$= \text{tr}\{P_{\lambda,i}(\rho_{AB}^{\otimes n})^{\otimes n}\} / \dim F_\lambda, \quad (56)$$

so by the preceeding

$$\text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}P_{\lambda,j}\rho_{AB}^{\otimes n}\} \leq \frac{1}{\dim(F_\lambda)} \text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}\} \frac{1}{\dim(F_\lambda)} \text{tr}\{P_{\lambda,j}\rho_{AB}^{\otimes n}\} \dim(F_\lambda) \quad (57)$$

$$\text{(by assumption)} \leq \text{tr}\{(P_\mu \otimes P_\nu)\rho_{AB}^{\otimes n}\} \text{tr}\{P_\lambda\rho_{AB}^{\otimes n}\} \dim(F_\lambda)^{-1} \quad (58)$$

$$\text{(since } P_\nu \leq \mathbb{1}_{\mathcal{H}_B^{\otimes l}}) \leq \text{tr}\{P_\mu\rho_A^{\otimes n}\} \text{tr}\{P_\lambda\rho_{AB}^{\otimes n}\} \dim(F_\lambda)^{-1} \quad (59)$$

$$\text{(Theorem 2, inequality 37)} \leq (n+1)d_A^2 2^{-nD(\mu||r_A)} (n+1)d_A^2 d_B^2 2^{-nD(\lambda||r)} 2^{-n(H(\bar{\lambda})-\gamma_1(n))} \quad (60)$$

$$\text{(Pinsker's inequality)} \leq (n+1)d_A^2 2^{-n2\delta^2} (n+1)d_A^2 d_B^2 2^{-nD(\lambda||r)} 2^{-n(H(\bar{\lambda})-\gamma_1(n))} \quad (61)$$

$$\text{(Lemma 1)} \leq 2^{-n2\delta^2} (n+1)2^{d_A^2 d_B^2} 2^{-nD(\lambda||r)} 2^{-n(H(r)-\gamma_1(n)-\varepsilon \log(d_A d_B/\varepsilon))} \quad (62)$$

$$= 2^{-n(2\delta^2+D(\lambda||r)+H(r)-\gamma_2(n)+\varepsilon \log(\varepsilon d_A d_B))} \quad (63)$$

$$\leq 2^{-n(2\delta^2+H(r)-\gamma_2(n)+\varepsilon \log(\varepsilon d_A d_B))}. \quad (64)$$

For the second statement, just read the proof until equation (57), then use the estimate given in Theorem 2, apply Pinsker's inequality and inequality 37. It follows

$$\text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}P_{\lambda,j}\rho_{AB}^{\otimes n}\} \leq \frac{1}{\dim(F_\lambda)} \text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}\} \frac{1}{\dim(F_\lambda)} \text{tr}\{P_{\lambda,j}\rho_{AB}^{\otimes n}\} \dim(F_\lambda) \quad (65)$$

$$\text{(Theorem 2, inequality 37)} \leq (n+1)2^{d_A^2 d_B^2} 2^{-n2D(\lambda||r)} 2^{-n(H(\bar{\lambda})-\gamma_1(n))} \quad (66)$$

$$\leq 2^{-n(4\varepsilon^2+H(\bar{\lambda})-\gamma_7(n))}. \quad (67)$$

□

Lemma 3. For a type $N(\cdot)$ on $[d_A d_B]^n$ and its corresponding typeclass $t \subset [d_A d_B]^n$ and $e_1, \dots, e_{d_A d_B}$ a basis in which $\rho_{AB}^{\otimes l}$ is diagonal, let $\mathcal{H}_t := \text{span}(\{e_{x_1} \otimes \dots \otimes e_{x_l} : x^l \in t\})$. Denote the projection onto \mathcal{H}_t by $p_{\mathcal{H}_t}$.

Then \mathcal{H}_t is invariant under the action \mathbb{B}^{AB} of S_n and for every λ with $\lambda = N^\downarrow$, $P'_{\lambda,i} \leq p_{\mathcal{H}_t}$ for at least one $i \in [m_\lambda^{AB}]$ (here, $P'_{\lambda,i}$ denotes the projection onto a copy of the irreducible subspace F_λ within $\mathcal{H}^{\otimes n}$).

Remark 3. $P'_{\lambda,i}$ is not necessarily equal to any of the $P_{\lambda,j}$ that were defined in the introduction.

Proof. To a given $N(\cdot)$, take T to be the standard tableaux for $\lambda = N^\downarrow$ which has entries $T_{1i} = i$, $T_{2i} = \lambda_1 + i$ and so on, until finally $T_{\lambda_{d_A d_B} i} = \lambda_1 + \dots + \lambda_{n-1} + i$. Let $v = \otimes_{i=1}^{d_A d_B} e_i^{\otimes N(i)}$. Denote the set of row permutations belonging to T by R_T , the column permutations by C_T and set $E_T := \{\pi \circ \tau : \pi \in C_T, \tau \in R_T\}$. Note that $V_\lambda := \text{span}(\{E(T)v : v \in \mathcal{H}^{\otimes n}, T \text{ - standard tableaux for } \lambda\})$ is the isotypical vectorspace belonging to λ - it holds $\text{supp}(P_\lambda) = V_\lambda$. We calculate the overlap of v with a suitably chosen element of V_λ :

$$\langle v, \mathbb{B}(E_T)v \rangle = \sum_{\pi \in C_T} \text{sgn}(\pi) \sum_{\tau \in R_T} \langle \mathbb{B}(\pi)\mathbb{B}(\tau)v, v \rangle \quad (68)$$

$$= |R_T| \sum_{\pi \in C_T} \text{sgn}(\pi) \langle \mathbb{B}(\pi)v, v \rangle \quad (69)$$

$$= |R_T|, \quad (70)$$

since $\langle \mathbb{B}(\pi)v, v \rangle = 0$ for every $C_T \ni \pi \neq e$. Now assume that \mathcal{H}_t contains no irreducible subspace corresponding to λ . Then, of course, for every vector $w \in \text{supp}(P_\lambda)$ we have $w \perp \mathcal{H}_t$. But by the preceding, the vector $w := \mathbb{B}(E_T)v \in \text{supp}(P_\lambda)$ is not perpendicular to \mathcal{H}_t . Thus, there must be at least one copy of F_λ in \mathcal{H}_t , which is what we set out to prove. \square

Lemma 4. For any $\lambda \in YF_{d_A d_B, n}$, it holds that

$$\dim(F_\lambda) 2^{n^2 \sum_{i=1}^{d_A d_B} \bar{\lambda}_i \log r_i} \leq \text{tr}\{P_\lambda(\rho_{AB}^2)^{\otimes n}\}. \quad (71)$$

In case that $r_i = 0$ holds for an $i \in [d_A d_B]$, we again use the convention $2^{-\infty} = 0$, so that the formula is valid also in this case.

Remark 4. This result can, with a subexponentially different lower bound, also be found in [4, equation (6.20)].

Proof. By Lemma 3, for the subspace \mathcal{H}_t defined by the typeclass corresponding to λ , we have $P_{\lambda,i} \leq p_{\mathcal{H}_t}$ for at least one $i \in [m_\lambda^{AB}]$. Also, $\langle v, (\rho_{AB}^2)^{\otimes n} v \rangle = \prod_{j=1}^{d_A d_B} r_j^{2\lambda_j}$ for every $v \in \mathcal{H}_t$. Thus,

$$\text{tr}\{(P_\lambda \rho_{AB}^{\otimes n})^2\} = \text{tr}\{P_\lambda(\rho_{AB}^2)^{\otimes n}\} \quad (72)$$

$$\geq \text{tr}\{P'_{\lambda,i}(\rho_{AB}^2)^{\otimes n}\} \quad (73)$$

$$= \text{tr}\{P'_{\lambda,i} p_{\mathcal{H}_t}(\rho_{AB}^2)^{\otimes n} p_{\mathcal{H}_t}\} \quad (74)$$

$$= \prod_{j=1}^{d_A d_B} r_j^{2\lambda_j} \text{tr}\{P'_{\lambda,i} p_{\mathcal{H}_t}\} \quad (75)$$

$$= 2^{n^2 \sum_{j=1}^{d_A d_B} \bar{\lambda}_j \log r_j} \dim(F_\lambda). \quad (76)$$

\square

Corollary 1. It is an immediate consequence of Lemma 4 and Lemma 1 that for any $\varepsilon > 0$ we have

$$\text{tr}\{P_\varepsilon^{AB} \rho_{AB}^{\otimes n} P_\varepsilon^{AB} \rho_{AB}^{\otimes n}\} \geq 2^{-n(H(r) + 2\varepsilon c_1 + \varepsilon \log(d_A d_B / \varepsilon))} \quad (77)$$

with the constant $c_1 = c_1(r)$ given by $c_1 := -\log \min\{r_i : r_i > 0, i \in [d_A d_B]\}$.

Proof. (Of Theorem 1): We first prove the promised lower bounds. Doing so, we use the definition $B_\varepsilon(r_X) := \{p \in \mathcal{P}([\dim X]) : \|p - r_X\| \leq \varepsilon\}$, $X \in \{A, B, AB\}$. It obviously holds

$$\text{tr}\{[(P_\varepsilon^A \otimes P_\varepsilon^B) \rho_{AB}^{\otimes n}]^2\} \geq \text{tr}\{[(P_\varepsilon^A \otimes P_\varepsilon^B) P_\delta^{AB} \rho_{AB}^{\otimes n}]^2\}. \quad (78)$$

Now, set

$$X := \text{tr}\{P_\delta^{AB}(\rho_{AB}^2)^{\otimes n}\} - \text{tr}\{[(P_\varepsilon^A \otimes P_\varepsilon^B)P_\delta^{AB}\rho_{AB}^{\otimes n}]^2\}. \quad (79)$$

It is clear that $X \geq 0$ holds. Define the sets

$$\mathcal{A} := \{P_{\lambda,i} : \lambda \in B_\delta(r), P_{\lambda,i} \leq P_\mu \otimes P_\nu \text{ for } \mu, \nu \text{ with } \|\bar{\mu} - r_A\| > \varepsilon, \|\bar{\nu} - r_B\| \leq \varepsilon\} \quad (80)$$

$$\mathcal{B} := \{P_{\lambda,i} : \lambda \in B_\delta(r), P_{\lambda,i} \leq P_\mu \otimes P_\nu \text{ for } \mu, \nu \text{ with } \|\bar{\mu} - r_A\| \leq \varepsilon, \|\bar{\nu} - r_B\| > \varepsilon\} \quad (81)$$

$$\mathcal{AB} := \{P_{\lambda,i} : \lambda \in B_\delta(r), P_{\lambda,i} \leq P_\mu \otimes P_\nu \text{ for } \mu, \nu \text{ with } \|\bar{\mu} - r_A\| > \varepsilon, \|\bar{\nu} - r_B\| > \varepsilon\} \quad (82)$$

$$\mathcal{E} := \mathcal{A} \cup \mathcal{B} \cup \mathcal{AB}. \quad (83)$$

Then according to Lemma 2 for $P_{\lambda,i} \in \mathcal{E}$ and $P_{\lambda,j}$ with $\lambda \in B_\delta(r)$ the estimate

$$\text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}P_{\lambda,j}\rho_{AB}^{\otimes n}\} \leq 2^{-n(2\varepsilon^2+H(r)+\delta\log(\delta d_A d_B)-\gamma_2(n))} \quad (84)$$

holds and, therefore,

$$X = \sum_{\lambda \in B_\delta(r)} \sum_{j=1}^{m_\lambda^{AB}} \left(\sum_{P_{\lambda,i} \in \mathcal{E}} \text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}P_{\lambda,j}\rho_{AB}^{\otimes n}\} + \sum_{P_{\lambda,i} \in \mathcal{E}} \text{tr}\{P_{\lambda,j}\rho_{AB}^{\otimes n}P_{\lambda,i}\rho_{AB}^{\otimes n}\} \right) \quad (85)$$

$$= 2 \sum_{\lambda \in B_\delta(r)} \sum_{j=1}^{m_\lambda^{AB}} \sum_{P_{\lambda,i} \in \mathcal{E}} \text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}P_{\lambda,j}\rho_{AB}^{\otimes n}\} \quad (86)$$

$$\leq 2 \sum_{\lambda \in B_\delta(r)} \sum_{j=1}^{m_\lambda^{AB}} \sum_{P_{\lambda,i} \in \mathcal{E}} 2^{-n(2\varepsilon^2+H(r)+\delta\log(\delta d_A d_B)-\gamma_2(n))} \quad (87)$$

$$\leq 2 \cdot |B_\delta(r)| \cdot (m_\lambda^{AB})^2 \cdot 2^{-n(2\varepsilon^2+H(r)+\delta\log(\delta d_A d_B)-\gamma_2(n))} \quad (88)$$

$$\leq 2 \cdot (n+1)^{d_A d_B} \cdot (n+1)^{(d_A d_B)^4} \cdot 2^{-n(2\varepsilon^2+H(r)+\delta\log(\delta d_A d_B)-\gamma_2(n))} \quad (89)$$

$$= 2^{-n(2\varepsilon^2+H(r)+\delta\log(\delta d_A d_B)-\gamma_3(n))}, \quad (90)$$

where the last inequality follows from equation (1.22) in [2] and the type counting bound Lemma 2.2 in [1] and the function γ_3 is defined by $\gamma_3(n) := \frac{(d_A d_B)^5}{n} \log(2(n+1)) + \gamma_2(n)$ and satisfies $\lim_{n \rightarrow \infty} \gamma_3(n) = 0$.

We arrive at

$$\text{tr}\{[(P_\varepsilon^A \otimes P_\varepsilon^B)\rho_{AB}^{\otimes n}]^2\} \geq \text{tr}\{P_\delta^{AB}(\rho_{AB}^2)^{\otimes n}\} - X \quad (91)$$

$$\geq 2^{-n(H(r)+2\delta c_1+\delta\log(d_A d_B/\delta))} - 2^{-n(2\varepsilon^2+H(r)+\delta\log(\delta/d_A d_B)-\gamma_3(n))} \quad (92)$$

$$= 2^{-nH(r)}(2^{-n(2\delta c_1+\delta\log(d_A d_B/\delta))} - 2^{-n(2\varepsilon^2+\delta\log(\delta/d_A d_B)-\gamma_3(n))}). \quad (93)$$

By reading only the r.h.s. of the above inequalities and remembering the definition of X , this proves the l.h.s. estimate of inequality (13) of Theorem 1 with $\varphi(\delta) := 2\delta c_1 + \delta\log(d_A d_B/\delta)$ and for every function $\gamma : \mathbb{N} \mapsto \mathbb{R}_+$ with $\gamma \geq \gamma_3$.

To show inequality (16), choose $\delta = d_A d_B/n$, define $\gamma_4(n) := 2c_1 d_A d_B/n + d_A d_B \log(n)/n$, $\gamma_5(n) := -d_A d_B \log(n)/n + \gamma_3(n)$ and $\gamma_6(n) := \max\{\gamma_4(n), \gamma_5(n)\}$.

Then our estimate reads

$$\text{tr}\{[(P_\varepsilon^A \otimes P_\varepsilon^B)\rho_{AB}^{\otimes n}]^2\} \geq 2^{-nH(r)}(2^{-n(2\gamma_4(n))} - 2^{-n(2\varepsilon^2-\gamma_5(n))}) \quad (94)$$

$$\geq 2^{-nH(r)}2^{-n\gamma_6(n)}(1 - 2^{-n2\varepsilon^2}). \quad (95)$$

This shows (16) for every function $\gamma : \mathbb{N} \mapsto \mathbb{R}_+$ with $\gamma \geq \gamma_6$. The r.h.s. inequality of (13) can be obtained by the estimate $P_\varepsilon^A \otimes P_\varepsilon^B \leq \mathbb{1}_{\mathcal{H}_{AB}^{\otimes n}}$, followed by application of equation (57), the estimate

$\text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}\} \leq 1$, equation (29) and Lemma 1.

Inequality (14) follows by application of Theorem 2 (see, for example [2], Corollaries 2.14 and 2.15). More precisely, this inequality will, again, be valid for all functions $\gamma : \mathbb{N} \mapsto \mathbb{R}_+$ with $\gamma \geq \gamma_7$, where γ_7 is a nonnegative function on the natural numbers which vanishes for n going to infinity, as do all the other functions γ_i , ($i \in \mathbb{N}$).

Inequality (17) follows from (14) by choosing $\delta > \varepsilon$.

The inequalities (12) and (15) both follow from Winters gentle measurement lemma [10].

Inequality (11) can be obtained using the following chain of inequalities:

$$\text{tr}\{\text{tr}_B\{\Phi\}^2\} \leq \text{tr}\{\text{tr}_B\{(P_\varepsilon^A \otimes P_\varepsilon^B)\rho_{AB}^{\otimes n}(P_\varepsilon^A \otimes P_\varepsilon^B)\}^2\} \quad (96)$$

$$\leq \text{tr}\{P_\varepsilon^A \text{tr}_B\{(\mathbb{1}_A^{\otimes n} \otimes P_\varepsilon^B)\rho_{AB}^{\otimes n}(\mathbb{1}_A^{\otimes n} \otimes P_\varepsilon^B)\}P_\varepsilon^A \rho_A^{\otimes n}\} \quad (97)$$

$$\leq \text{tr}\{P_\varepsilon^A \rho_A^{\otimes n} P_\varepsilon^A \rho_A^{\otimes n}\} \quad (98)$$

$$\leq 2^{-n(H(r_A) - \gamma(n))}, \quad (99)$$

where, at last, γ is defined via $\gamma(n) := \max\{\gamma_i(n)\}_{i=1}^7$ and the r.h.s. estimate of inequality (13) is used. Note that inequality (96) is valid because P_δ^{AB} commutes with $\rho_{AB}^{\otimes n}$, while the step from (96) to (97) is possible because of the estimate $\text{tr}_B\{(\mathbb{1}_A \otimes P^B)X_{AB}(\mathbb{1}_A \otimes P^B)\} \leq \text{tr}_B\{X_{AB}\}$, which is valid for any nonnegative operator X_{AB} on a composite system AB and projection P^B on the B -part of the system. \square

5 Conclusion and Outlook

We proved 2-party typicality by application of representation theory of the symmetric group. Most of the work we did involved the l.h.s. estimate of (13), an estimate which was not needed to prove 2-party typicality. Indeed, the necessary estimates for two-party typicality follow almost trivially from the fact that $(P_\varepsilon^A \otimes P_\varepsilon^B)P_\delta^{AB}$ is a projection, which in turn is due to the structure of the representation of involved groups (see Remark 1).

Despite several attempts we were not able to generalize our result to 3-party typicality. We suspect that this is due to *noncommutativity*: Extending our notation to tripartite systems ABC in the obvious way, we believe that the operators $P_\lambda^{AB} \otimes P_\nu^C$ and $P_{\lambda'}^A \otimes P_{\nu'}^{BC}$ do not commute in general. This, in turn, forbids a standard application of the gentle measurement Lemma.

We hope that future research will enable us to give a more precise formulation of the multiparty typicality conjecture in terms of representation theoretic objects.

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